

On Integrable modules for the twisted full toroidal Lie algebra

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Abstract

The paper is to classify irreducible integrable modules for the twisted full toroidal Lie algebra with some technical conditions. The twisted full toroidal Lie algebra are extensions of multiloop algebra twisted by several finite order automorphisms. This result generalizes a result by Fu Jiayuan and Cuipo Jiang $[FJ]$, where they consider only one automorphism.

Key words : Multiloop algebras, Finite order automorphism, integrable modules.

MSC : Primary 17B67, Secondary 17B65, 17B70.

Introduction

The main purpose of this paper is to classify irreducible integrable modules for the twisted full toroidal Lie algebra under certain technical conditions. The twisted full toroidal Lie algebra in several variables is defined using several commuting finite order automorphisms of the underlying finite dimensional simple Lie algebra \mathfrak{g} . The result of this paper generalizes the main theorem of $[FJ]$, where they consider only one automorphism.

The twisted full toroidal Lie algebra is a natural generalization of the classical twisted affine Lie algebra. The classical procedure of realizing twisted affine Lie algebras using loop algebra in one variable proceeds in two steps $[K]$. In the first step, the derived algebra modulo its center of the affine Lie

algebra is constructed as the algebra of a diagram automorphism of a finite dimensional simple Lie algebra. In the second step the affine Lie algebra is built from the graded loop algebra by forming universal central extension (one dimensional center) and adding graded algebra derivations (are dimensional).

The replacement of derived Lie algebra is multiloop algebra (Sec. 1.5) in several variables twisted by finitely many automorphisms. Then we consider the Universal central extension of the multiloop algebra (infinite dimensional) and add graded algebra of derivations (infinite dimensional) which we call twisted full toroidal Lie algebra and denoted by τ .

In this paper we classify irreducible integrable modules for τ with finite dimensional weight spaces with non-zero central action and with some technical conditions that are satisfied for the well known Lie algebras called Lie Torus. See [ABFP].

The contents of the paper are the following. We fix an irreducible integrable module for τ with non-zero central action.

The central operators act as scalars. Upto choice of co-ordinates (in other words upto an automorphism) we can assume that K_0 acts as $C_0 > 0$ and K_i acts trivially for $i \neq 0$. In the first main Theorem 5.3 we prove that V is an highest weight module with some natural triangular decomposition

$$\tau = \tau^- \oplus \tau^0 \oplus \tau^+.$$

Let T be the highest weight space which is an irreducible τ_0 -module (Proposition 6.2) and naturally \mathbb{Z}^n - graded. In Sections 7, 8 and 9, we describe T as τ_0 -module. We noticed that some parts of τ_0 acts as scalars and hence we only consider a subalgebra L (See 7.3) for which T is irreducible. We also consider a certain subalgebra \tilde{L} of L (See 8.1) and consider a certain finite dimensional quotient \tilde{V} of T , which is \tilde{L} -module. Our approach is to describe \tilde{V} as \tilde{L} -module and obtain T as L -module.

In the process we define an L -module $L(\tilde{V})$ (See 8.4) and prove $L(\tilde{V})$ naturally decomposes into L -modules (See notes after Lemma 8.9 and 8.10). We prove one of the component is isomorphic to T . We also prove in

Theorem 8.17 that \tilde{V} is irreducible \tilde{L} -module if and only if $L(\tilde{V})$ decomposes into mutually non-isomorphic irreducible L -modules. In this case T can be identified in a natural way as a component of $L(\tilde{V})$. In the case \tilde{V} is reducible, we prove that \tilde{V} is completely reducible as \tilde{L} -module and all components are isomorphic (See Proposition 8.27). In this case also T is a submodule of $L(\tilde{V})$ but the inclusion is twisted. In the rest of the paper we describe each component of \tilde{V} . It turns out to be an irreducible module for $gl_n \oplus \mathfrak{g}$ (Theorem 9.4). See 7.5 for the Definition of \mathfrak{g} . We will indicate in 9.6, given an irreducible module for $gl_n \oplus \mathfrak{g}$, how to obtain a module for L and thereby for τ_0 . In the Theorem 9.7 we will state our final result that the original module V is an irreducible quotient of an induced module of T .

1 Notation and Preliminaries

Throughout this paper we will use the following notation

- (1.1) All vector spaces, algebras and tensor products are over complex numbers \mathbb{C} . Let \mathbb{Z}, \mathbb{N} and \mathbb{Z}_+ denote integers, non-negative integers and positive integers.
- (1.2) Let \mathfrak{g} be a finite dimensional simple Lie algebra and let $(,)$ be a non-degenerate symmetric bilinear form on \mathfrak{g} . We fix a positive integer n . Let $\sigma_0, \sigma_1, \dots, \sigma_n$ be commuting finite order automorphisms of \mathfrak{g} of order m_0, m_1, \dots, m_n respectively. Let $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$. Let $k = (k_1, \dots, k_n)$ and $l = (l_1, \dots, l_n)$ denote vectors in \mathbb{Z}^n .
- (1.3) Let $\Gamma = m_1\mathbb{Z} \oplus \dots \oplus m_n\mathbb{Z}$ and $\Gamma_0 = m_0\mathbb{Z}$. Let $\Lambda = \mathbb{Z}^n / \Gamma$ and $\Lambda_0 = \mathbb{Z} / \Gamma_0$. Let \bar{k} and \bar{l} denote the images in Λ . For any integers k_0 and l_0 , let \bar{k}_0 and \bar{l}_0 denote the images in Λ_0 .

Let

$$\begin{aligned}
A &= \mathbb{C}[t_0^{\pm 1}, \dots, t_n^{\pm 1}], \\
A_n &= \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \\
A(m) &= \mathbb{C}[t_1^{\pm m_1}, \dots, t_n^{\pm m_n}], \\
A(m_0, m) &= \mathbb{C}[t_0^{\pm m_0}, t_1^{\pm m_1}, \dots, t_n^{\pm m_n}].
\end{aligned}$$

(1.4) For $k \in \mathbb{Z}^n$, let $t^k = t_1^{k_1} \dots t_n^{k_n} \in A_n$. Let ΩA be the vector space spanned by symbols $t_0^{k_0} t^k K_i, 0 \leq i \leq n, k_0 \in \mathbb{Z}, k \in \mathbb{Z}^n$. Let dA be the subspace spanned by $\sum_{i=0}^n k_i t_0^{k_0} t^k K_i$.

Let $L(\mathfrak{g}) = \mathfrak{g} \otimes A$ and notice that it has a natural structure of a Lie algebra. We will now define toroidal Lie algebra

$$\tilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \Omega A / dA.$$

Let $X(k_0, k) = X \otimes t_0^{k_0} t^k$ and $Y = Y \otimes t_0^{l_0} t^l$ for $X, Y \in \mathfrak{g}, k_0, l_0 \in \mathbb{Z}$ and $k, l \in \mathbb{Z}^n$.

$$(1.4.1) \quad [X(k_0, k), Y(l_0, l)] = [X, Y](l_0 + k_0, l + k) + (X, Y) \sum k_i t_0^{l_0 + k_0} t^{l+k} K_i.$$

(1.4.2) $\Omega A / dA$ is central.

It is well known that $\tilde{L}(\mathfrak{g})$ is the universal central extension of $L(\mathfrak{g})$. See [EMY] and [Ka].

(1.5) We will now define multiloop algebra as a subalgebra of $L(\mathfrak{g})$. See [ABFB] for more details. For $0 \leq i \leq n$, let ξ_i be a m_i th primitive root of unity.

Let

$$\mathfrak{g}(\bar{k}_0, \bar{k}) = \{x \in \mathfrak{g} | \sigma_i x = \xi_i^{k_i} x, 0 \leq i \leq n\}.$$

Then define

$$L(\mathfrak{g}, \sigma) = \bigoplus_{(k_0, k) \in \mathbb{Z}^{n+1}} \mathfrak{g}(\bar{k}_0, \bar{k}) \otimes t_0^{k_0} t^k,$$

which is called a multiloop algebra.

(1.6) The finite dimensional irreducible modules for $L(\mathfrak{g}, \sigma)$ are classified by Michael Lau [ML].

(1.7) Suppose \mathfrak{h}_1 is a finite dimensional ad-diagonalizable subalgebra of a Lie algebra \mathfrak{g}_1 . We set for $\alpha \in \mathfrak{h}_1^*$

$$\mathfrak{g}_{1,\alpha} = \{x \in \mathfrak{g}_1 \mid [h.x] = \alpha(h)x, h \in \mathfrak{h}_1\}.$$

Then we have

$$\mathfrak{g}_1 = \bigoplus_{\alpha \in \mathfrak{h}_1^*} \mathfrak{g}_{1,\alpha}.$$

Let $\Delta(\mathfrak{g}_1, \mathfrak{h}_1) = \{\alpha \in \mathfrak{h}_1^* \mid \mathfrak{g}_{1,\alpha} \neq 0\}$, which includes zero.

Let $\Delta^\times(\mathfrak{g}_1, \mathfrak{h}_1) = \Delta(\mathfrak{g}_1, \mathfrak{h}_1) \setminus \{0\}$.

(1.8) We will now define the universal central extension of $L(\mathfrak{g}, \sigma)$. Define $\Omega A(m_0, m)$ and $dA(m_0, m)$ similar to the definition of ΩA and dA by replacing A by $A(m_0, m)$. Denote $Z(m_0, m) = \Omega A(m_0, m)/dA(m_0, m)$ and note that $Z(m_0, m) \subseteq \Omega A/dA$.

Define

$$\tilde{L}(\mathfrak{g}, \sigma) = L(\mathfrak{g}, \sigma) \oplus Z(m_0, m).$$

Let $X \in \mathfrak{g}(\bar{k}_0, \bar{k})$ and $Y \in \mathfrak{g}(\bar{l}_0, \bar{l})$ and let $X(k_0, k) = X \otimes t_0^{k_0} t^k$ and $Y(l_0, l) = Y \otimes t_0^{l_0} t^l$. Define

$$(1.8.1) \quad [X(k_0, k), Y(l_0, l)] = [X, Y](k_0 + l_0, k + l) + (X, Y) \sum k_i t_0^{l_0 + k_0} t^{l+k} K_i.$$

$$(1.8.2) \quad Z(m_0, m) \text{ is central.}$$

Notice that $(X, Y) \neq 0 \Rightarrow k + l \in \Gamma$ and $k_0 + l_0 \in \Gamma_0$. This follows from the standard fact that $(,)$ is invariant under σ_i for $0 \leq i \leq n$. This proves that the above Lie bracket is well defined. This Lie bracket is nothing but the restriction defined in (1.4).

(1.9) **Proposition** $\tilde{L}(\mathfrak{g}, \sigma)$ is the universal central extension of $L(\mathfrak{g}, \sigma)$. See Corollary (3.27) of [JS].

2 Derivation algebra of $A(m_0, m)$ and its extension to $Z(m_0, m)$.

(2.1) Let $D(m_0, m)$ be the derivation algebra of $A(m_0, m)$. From now onwards we let s and r to be in Γ and s_0 and r_0 to be in Γ_0 .

For $0 \leq i \leq n$ define $D_i(s_0, s) = t_0^{s_0} t^s t_i \frac{d}{dt_i}$ which acts on $A(m_0, m)$ as derivations. It is well known that $D(m_0, m)$ has the following basis

$$\{D_i(s_0, s) | 0 \leq i \leq n, s_0 \in \Gamma_0, s \in \Gamma\}.$$

Let $d_i = t_i \frac{d}{dt_i}$ and it is easy to see that

$$(2.1.1) \quad [t_0^{s_0} t^s d_a, t_0^{r_0} t^r d_b] = r_a t_0^{r_0+s_0} t^{r+s} d_b - s_b t_0^{r_0+s_0} t^{r+s} d_a.$$

(2.2) $D(m_0, m)$ acts on $Z(m_0, m)$ in the following way

$$(2.2.1) \quad t_0^{s_0} t^s d_a \cdot (t_0^{r_0} t^r K_b) = r_a t_0^{r_0+s_0} t^{r+s} K_b + \delta_{ab} \sum_{p=0}^n s_p t_0^{r_0+s_0} t^{r+s} K_p.$$

(2.3) It is known that $D(m_0, m)$ admits two non-trivial 2-cocycles with values in $Z(m_0, m)$. See [BB] for details

$$\begin{aligned} \varphi_1(t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b) &= -s_a r_b \sum_{p=0}^n r_p t_0^{r_0+s_0} t^{r+s} K_p, \\ \varphi_2(t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b) &= r_a s_b \sum_{p=0}^n r_p t_0^{r_0+s_0} t^{r+s} K_p. \end{aligned}$$

(2.4) Let φ be arbitrary linear combinations of φ_1 and φ_2 . Then there is a corresponding Lie algebra

$$(2.4.1) \quad \tau = L(\mathfrak{g}, \sigma) \oplus Z(m_0, m) \oplus D(m_0, m).$$

The Lie brackets are defined in the following way in addition to 1.8.1 and 1.8.2.

$$(2.4.2) \quad [t_0^{r_0} t^r d_a, X(k_0, k)] = k_a X(k_0 + r_0, k + r),$$

$$(2.4.3) \quad [t_0^{r_0} t^r d_a, t_0^{s_0} t^s K_b] = s_a t_0^{r_0+s_0} t^{r+s} K_b + \delta_{ab} \sum_{p=0}^n r_p t_0^{r_0+s_0} t^{r+s} K_p,$$

$$(2.4.4) \quad [t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b] = s_a t_0^{r_0+s_0} t^{r+s} d_b - r_b t_0^{r_0+s_0} t^{r+s} d_a + \varphi(t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b),$$

where $r, s \in \Gamma, r_0, s_0 \in \Gamma_0, X \in \mathfrak{g}(\bar{k}_0, \bar{k})$.

3 Assumptions and automorphisms.

In this section we will make some assumptions on $L(\mathfrak{g}, \sigma)$ which will hold throughout this paper. We will also define a class of automorphisms on τ .

(3.1) Assumptions

(3.1.1) $\mathfrak{g}(\overline{\circ}, \overline{\circ})$ is simple Lie algebra.

(3.1.2) We can choose Cartan subalgebra $\mathfrak{h}(\circ)$ and \mathfrak{h} for $\mathfrak{g}(\overline{\circ}, \overline{\circ})$ and \mathfrak{g} such that $\mathfrak{h}(\circ) \subseteq \mathfrak{h}$.

(3.1.3) It is known that $\Delta_0^\times = \Delta(\mathfrak{g}(\overline{\circ}, \overline{\circ}), \mathfrak{h}(\circ)) \setminus \{0\}$ is an irreducible reduced finite root system and has atmost two root lengths. Let $\Delta_{0,sh}^\times$ be the set of non-zero short roots. Define

$$\Delta_{0,en}^\times = \begin{cases} \Delta_0^\times \cup 2\Delta_{0,sh}^\times & \text{if } \Delta_0^\times \text{ is of type } B_l \\ \Delta_0^\times & \text{otherwise} \end{cases}$$

$$\Delta_{0,en} = \Delta_{0,en}^\times \cup \{0\}.$$

We assume that $\Delta(\mathfrak{g}, \mathfrak{h}(0)) = \Delta_{0,en}$.

(3.2) **Remark** These assumptions are true for any Lie Torus. See Proposition 3.2.5 of [ABFP].

It should be mentioned that Lie Torus are very important class of Lie algebras and give rise to almost all Extended Affine Lie algebras. See [ABFP] and references there in.

(3.3) Change of co-ordinates

In this subsection we will define a class of automorphisms for the Lie algebra

$$\tau(1, \mathbf{1}) = \mathfrak{g} \otimes A \oplus \Omega_A/d_A \oplus D(1, 1).$$

It is standard fact that $GL(n+1, \mathbb{Z})$ acts on \mathbb{Z}^{n+1} and we denote the action by dot. Let $B = (b_{ij}) \in GL(n+1, \mathbb{Z})$, then define automorphism, again denote by B on $\tau(1, \mathbf{1})$.

Let $t_0^{k_0} t^k = t(k_0, k)$, then define

$$\begin{aligned} B \cdot x \otimes t(k_0, k) &= x \otimes t^{B \cdot (k_0, k)}, \\ B \cdot t(k_0, k) K_j &= \sum_{p=0}^n b_{pj} t^{B \cdot (k_0, k)} K_p, \\ B \cdot t(k_0, k) d_j &= \sum_{p=0}^n c_{pj} t^{B \cdot (k_0, k)} d_p, \end{aligned}$$

where $B^{-1} = (c_{pj})$.

This is what we call change of co-ordinates. We will use this change of co-ordinates without any mention and just say "upto choice of co-ordinates"

4 Root space decomposition and integrable modules for τ .

(4.1) First note the center of τ is spanned by K_0, K_1, \dots, K_n .

Let $H = \mathfrak{h}(0) \oplus \sum \mathbb{C} K_i \oplus \sum \mathbb{C} d_i$ which is an abelian Lie algebra of τ and plays the role of Cartan subalgebra.

Define $\delta_i, \Lambda_i \in H^* (0 \leq i \leq n)$ be such that

(4.1.1)

$$\begin{aligned} \Lambda_i(\mathfrak{h}(0)) &= 0, \Lambda_i(K_j) = \delta_{ij}, \Lambda_i(d_j) = 0, \\ \delta_i(\mathfrak{h}(0)) &= 0, \delta_i(K_j) = 0, \delta_i(d_j) = \delta_{ij}. \end{aligned}$$

Let $\delta_k = \sum_{i=1}^n k_i \delta_i$ for $k \in \mathbb{Z}^n$.

(4.1.2) Let $\mathfrak{g}(\bar{k}_0, \bar{k}, \alpha) = \{x \in \mathfrak{g}(\bar{k}_0, \bar{k}) | [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}(0)\}$

then τ has a root space decomposition.

(4.1.3) $\tau = \bigoplus_{\beta \in \Delta} \tau_\beta$

where $\Delta \subseteq \{\alpha + k_0 \delta_0 + \delta_k, \alpha \in \Delta_{0, e_n}, k_0 \in \mathbb{Z}, k \in \mathbb{Z}^n\}$.

$$\begin{aligned} \tau_{\alpha + k_0 \delta_0 + \delta_k} &= \mathfrak{g}(\bar{k}_0, \bar{k}, \alpha) \otimes t_0^{k_0} t^k \text{ for } \alpha \neq 0, \\ \tau_{k_0 \delta_0 + \delta_k} &= \mathfrak{g}(\bar{k}_0, \bar{k}, 0) \otimes t_0^{k_0} t^k \oplus \bigoplus_{i=0}^n \mathbb{C} t_0^{k_0} t^k K_i \oplus \bigoplus_{i=0}^n \mathbb{C} t_0^{k_0} t^k d_i. \end{aligned}$$

Notice that $\tau_0 = H$.

(4.1.4) Now we will define a non-degenerate bilinear form on H^* . For $\alpha \in \mathfrak{h}(0)^*$

extended α to H by $\alpha(K_i) = \alpha(d_i) = 0, 0 \leq i \leq n$.

Let $(\mathfrak{h}(0), K_i) = 0 = (\mathfrak{h}(0), d_i)$,

$(\delta_k + \delta_{k_0}, \delta_l + \delta_{l_0}) = 0 = (\Lambda_k, \Lambda_p)$,

$(\delta_i, \Lambda_j) = \delta_{ij}$. The form on $\mathfrak{h}(0)$ is the restriction of the form $(,)$ on \mathfrak{g} .

(4.1.5) For $\gamma = \alpha + k_0\delta_0 + \delta_k$ is called real root if $\alpha \neq 0$ which is equivalent to

$(\gamma, \gamma) \neq 0$. Denote Δ^{re} be the set of real roots. For

$\alpha \in \Delta_{0,en}$, denote α^\vee the co-root of α .

Define $\gamma^\vee = \alpha^\vee + \frac{2}{(\alpha, \alpha)} \sum_{i=0}^n k_i K_i$ for γ real.

Then $\gamma(\gamma^\vee) = \alpha(\alpha^\vee) = 2$.

For γ real root, define reflection on H^* by

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma, \gamma \in H^*.$$

Let W be the Weyl group generated by $r_\gamma, \gamma \in \Delta^{re}$.

(4.2) A module V of τ is called integrable if

$$(4.2.1) \quad V = \bigoplus_{\lambda \in H^*} V_\lambda, V_\lambda = \{v \in V | hv = \lambda(h)v, h \in H\}, \dim V_\lambda < \infty,$$

$$(4.2.2) \quad \mathfrak{g}(\overline{k_0}, \overline{k}, \alpha) \otimes t_0^{k_0} t^k \text{ acts locally nilpotently on } V \text{ for } \alpha \neq 0.$$

Let $P(V) = \{\gamma \in H^* | V_\gamma \neq 0\}$.

The following Lemma is very standard.

(4.3) **Lemma** Suppose V is an irreducible integrable module for τ . Then

(4.3.1) $P(V)$ is W -invariant.

(4.3.2) $\dim V_\gamma = \dim V_{w\gamma}$ for all $w \in W$.

(4.3.3) For $\alpha \in \Delta^{re}, \lambda \in P(V)$ we have $\lambda(\alpha^\vee) \in \mathbb{Z}$.

(4.3.4) For $\alpha \in \Delta^{re}, \lambda \in P(V)$. If $\lambda(\alpha^\vee) > 0$ then $\lambda - \alpha \in P(V)$.

(4.3.5) For $\lambda \in P(V), \lambda(K_i)$ is a constant integer.

The purpose of this paper is to classify irreducible integrable modules for τ with non-zero central action.

For an irreducible integrable module with non zero central charge, we can assume that K_0 acts as $C_0 > 0$ and $K_i (i \neq 0)$ acts trivially upto a choice of co-ordinates. See (3.3).

5 Existence of highest weight

Throughout the rest of the paper we fix an irreducible integrable module for τ with K_0 acting as $C_0 > 0$ and $K_i (i \neq 0)$ acts trivially. Notice that for any $\lambda \in P(V)$, $\lambda(K_i) = C_i = 0$ for $1 \leq i \leq n$ and $\lambda(K_0) = C_0$. Given a $\lambda \in H^*$ let λ' denote the restriction to $\mathfrak{h}(0)$. Given a λ' in $\mathfrak{h}^*(0)$, extend to H by $\lambda'(K_i) = \lambda(d_i) = 0$. Then λ can be uniquely written as

$$(5.1) \quad \lambda = \lambda' + \sum_{i=0}^n \lambda(d_i) \delta_i + \sum_{i=0}^n \lambda(K_i) \Lambda_i.$$

For $\lambda \in P(V)$

$$\lambda = \lambda' + \lambda(d_0) \delta_0 + \sum_{i=1}^n \lambda(d_i) \delta_i + \lambda(K_0) \Lambda_0.$$

Put $\bar{\lambda} = \lambda' + \lambda(d_0) \delta_0 + \lambda(K_0) \Lambda_0$,

so that $\lambda = \bar{\lambda} + \sum_{i=1}^n \lambda(d_i) \delta_i$. Let $\alpha_0 = -\beta_0 + \delta_0$ where β_0 is maximal root in $\Delta_{0,en}$. Note that α_0 may not be root of τ .

Let $\alpha_1, \alpha_2, \dots, \alpha_p$ be a set of simple roots for $\Delta(g(\bar{\sigma}, \bar{\sigma}), \mathfrak{h}(0))$ and let $Q^+ = \bigoplus_{i=0}^p \mathbb{N} \alpha_i$. Define an ordering on H^* , $\lambda \leq \mu$ for $\lambda, \mu \in H^*$, if $\mu - \lambda \in Q^+$. Notice that in this case $\mu(d_i) = \lambda(d_i)$ for $1 \leq i \leq n$.

(5.2) **Proposition** Assumptions as above. Given a $\mu \in P(V)$ there exists a $\lambda \in P(V)$ such that $\mu \leq \lambda$ and $\lambda + \alpha \notin P(V)$ for all $\alpha > 0$, where λ is dominant integral.

Proof The proof of the Proposition follows from the results of Section 6 of [E4]. We will briefly sketch the proof. Suppose the Proposition is false. Then the conclusion of Proposition 6.5 of [E4] holds and it will lead to contradiction as explained in the proof of Theorem 6.1 of [E4]. Also note that the proof in [E4] is worked out only for $n = 0$ but

holds for any n . Here we need to observe in the construction of λ_i 's in Proposition 6.5 of [E4], the $\delta_k, k \in \mathbb{Z}^n$ does not appear. In Proposition (5.2) λ is clearly dominant.

(5.3) **Theorem** Let V be an irreducible integrable module for τ with K_0 acts as $C_0 > 0$ and K_i acts trivially. Then there exists $\gamma \in P(V)$ such that $\gamma + \beta + \delta_k \notin P(V)$ for any $\beta > 0$ and for any $k \in \mathbb{Z}^n$.

Proof Suppose the theorem is false. First some word about notation. The δ 's that occur below are always linear combinations of $\delta_1, \dots, \delta_n$. Fix $\lambda_1 \in P(V)$ and let $\delta(g) = \sum_{i=1}^n \lambda(d_i) \delta_i$. Note that $(\lambda_1, \delta_i) = 0$ for $1 \leq i \leq n$ which follows from unique expression 5.1. Choose μ_1 as in the Proposition (5.2). Then $\lambda_1 \leq \mu_1$ and $\mu_1 + \alpha \notin P(V)$ for $\alpha > 0$. By assumptions there exists a root $\beta_1 + \delta(1)$ with $\beta_1 > 0$ and $\delta(1) \neq 0$ such that

$$\lambda_2 = \mu_1 + \beta_1 + \delta(1) \in P(V).$$

Notice that $\lambda_2 = \bar{\lambda}_2 + \delta(g) + \delta(1)$ and $\bar{\lambda}_1 \leq \bar{\mu}_1 < \bar{\lambda}_2$.

By repeating the argument infinitely many times we get dominant integral weights $\mu_d \in P(V)$ and roots $\beta_d + \delta(d)$ such that $\beta_d > 0$ and $\delta(d) \neq 0$ such that

$$(5.3.1) \quad \lambda_{d+1} = \mu_d + \beta_d + \delta(d) \in P(V).$$

$$(5.3.2) \quad \mu_d + \beta \notin P(V) \text{ for all positive } \beta.$$

$$(5.3.3) \quad \lambda_d \leq \mu_d,$$

$$(5.3.4) \quad \bar{\lambda}_d \leq \bar{\mu}_d < \bar{\lambda}_{d+1} \leq \bar{\mu}_{d+1},$$

It is easy to see that

$$(5.3.5) \quad \mu_d = \bar{\mu}_d + \delta(g) + \delta'(d), \text{ where } \delta'(d) = \sum_{k=1}^d \delta(k).$$

We also have

$$(5.3.6) \quad \bar{\mu}_{d_1} < \bar{\mu}_{d_2} \text{ for } d_1 < d_2.$$

$$(5.3.7) \quad (\mu_d + \beta_d, \beta_d) > 0, \text{ as } \mu_d \text{ is dominant integral.}$$

(5.3.8) **Claim** $\mu_{d_1} < \mu_{d_2} - \delta'(d_2) + \delta'(d_1)$ for $d_1 < d_2$.

To see the claim note that from (5.3.5) it follows that

$$\mu_{d_1} - \delta(g) - \delta'(d_1) < \mu_{d_2} - \delta(g) - \delta'(d_2).$$

The inequality still holds if we add $\delta(g) + \delta'(d_1)$ both sides. The claim follows.

Now there exists $d_1 < d_2$ such that $\delta'(d_2) - \delta'(d_1) = \delta_s$ for some $s \in \Gamma$.

We know that $\beta_{d_2} + \delta(d_2) \in \Delta$. It follows from the definition of τ that

$$\beta_{d_2} + \delta(d_2) + \delta'(d_2) - \delta'(d_1) \in \Delta.$$

Now from (5.1) and (5.3.7) it follows that

$$(\mu_{d_2} + \beta_{d_2} + \delta(d_2), \beta_{d_2} + \delta(d_2) + \delta'(d_2) - \delta'(d_1)) > 0.$$

From (4.3.4) it follows that

$$\mu_{d_2} + \delta'(d_1) - \delta'(d_2) \in P(V).$$

But by claim it follows that

$$\mu_{d_1} < \mu_{d_2} + \delta'(d_1) - \delta'(d_2),$$

which is a contradiction to (5.3.2). This proves the Theorem.

6 Triangle decomposition

(6.1) We will now define triangular decomposition for τ . Let $Z = \Omega A/dA$.

Let

$$\begin{aligned}
L^+(\mathfrak{g}, \sigma) &= \bigoplus_{\alpha+k_0\delta_0>0} \mathfrak{g}(\bar{k}_0, \bar{k}, \alpha) \otimes t_0^{k_0} t^k, k \in \mathbb{Z}^n; \\
L^-(\mathfrak{g}, \sigma) &= \bigoplus_{\alpha+k_0\delta_0<0} \mathfrak{g}(\bar{k}_0, \bar{k}, \alpha) \otimes t_0^{k_0} t^k, k \in \mathbb{Z}^n; \\
L^0(\mathfrak{g}, \sigma) &= \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(\bar{0}, \bar{k}, 0) t^k; \\
D^+(m_0, m) &= \bigoplus_{\substack{0 \leq i \leq n \\ s_0 > 0}} \mathbb{C} t_0^{s_0} t^s d_i, s \in \Gamma; \\
D^-(m_0, m) &= \bigoplus_{\substack{0 \leq i \leq n \\ s_0 < 0}} \mathbb{C} t_0^{s_0} t^s d_i, s \in \Gamma; \\
D^0(m_0, m) &= \bigoplus_{0 \leq i \leq n} \mathbb{C} t^s d_i, s \in \Gamma; \\
Z^+ &= \bigoplus_{\substack{0 \leq i \leq n \\ s_0 > 0}} \mathbb{C} t_0^{s_0} t^s K_i, s \in \Gamma; \\
Z^- &= \bigoplus_{\substack{0 \leq i \leq n \\ s_0 < 0}} \mathbb{C} t_0^{s_0} t^s K_i, s \in \Gamma; \\
Z^0 &= \bigoplus_{0 \leq i \leq n} \mathbb{C} t^s K_i, s \in \Gamma; \\
\tau^+ &= L^+(\mathfrak{g}, \sigma) \oplus Z^+ \oplus D^+(m_0, m); \\
\tau^- &= L^-(\mathfrak{g}, \sigma) \oplus Z^- \oplus D^-(m_0, m); \\
\tau^0 &= L^0(\mathfrak{g}, \sigma) \oplus Z^0 \oplus D^0(m_0, m).
\end{aligned}$$

Then clearly $\tau = \tau^- \oplus \tau^0 \oplus \tau^+$ is a triangular decomposition.

Let $T = \{v \in V | \tau^+ v = 0\} \neq 0$ by Theorem 5.3

(6.2) **Proposition:** T is a τ^0 - module and in fact irreducible as τ^0 - module.

Further $V = U(\tau^-)T$.

Proof It is easy to check that $[\tau^0, \tau^+] \subset \tau^+$. From this it follows that T is a τ^0 - module. Now from PBW theorem, we have $U(\tau) = U(\tau^-)U(\tau^0)U(\tau^+)$. (Here U denotes the universal enveloping algebra.) Using this and the fact that V is τ -irreducible, it follows that T is τ^0 -irreducible and $V = U(\tau^-)T$. Recall that $\{d_1, \dots, d_n\} \subseteq D^0(m_0, m)$ and hence T is \mathbb{Z}^n - graded.

Let $T_k = \{v \in T | d_i v = (\lambda(d_i) + k_i)v, 1 \leq i \leq n\}$

where λ is a fixed weight in $P(V)$. We will now record some result on T which can be proved similarly as in $[JM]$, $[EJ]$ and $[FJ]$.

(6.3) **Lemma**

$$(6.3.1) \text{ For } v \in T \setminus \{0\}, t^s K_0 v \neq 0 \text{ for all } s \in \Gamma.$$

$$(6.3.2) \text{ } \dim T_k = \dim T_{k+s} = d_k \quad \forall s \in \Gamma.$$

$$(6.3.3) \text{ Let } v_1(k), v_2(k), \dots, v_{d_k}(k) \text{ be a basis of } T_k. \text{ Let } v_i(s+k) = \frac{1}{C_0} t^s K_0 v_i(k), \\ \text{then } \{v_1(s+k), \dots, v_{d_k}(s+k)\} \text{ is a basis of } T_{k+s}.$$

$$(6.3.4) \text{ } \frac{1}{C_0} t^s K_0 (v_1(k+r), \dots, v_{d_k}(k+r)) = (v_1(k+r+s), \dots, v_{d_k}(k+r+s)) \\ \text{for all } r, s \in \Gamma.$$

$$(6.3.5) \text{ } h \otimes t^s (v_1(k+r), \dots, v_{d_k}(k+r)) = \lambda(h) (v_1(k+r+s), \dots, v_{d_k}(k+r+s)) \\ \text{for } h \in \mathfrak{h}(0) \text{ and all } r, s \in \Gamma, \text{ where } \lambda \text{ is a fixed weight of } P(V).$$

$$(6.3.6) \text{ } t^s d_0 (v_1(k+r), \dots, v_{d_k}(k+r)) = \lambda(d_0) (v_1(k+r+s), \dots, v_{d_k}(k+r+s)) \\ \text{for all } s, r \in \Gamma \text{ and for a fixed } \lambda \in P(V).$$

$$(6.3.7) \text{ } t^r K_p \cdot T = 0 \quad 1 \leq p \leq n, r \in \Gamma.$$

$$(6.3.8) \text{ } t^r K_0 \cdot t^s K_0 \cdot v = C_0 t^{r+s} K_0 v \quad \forall v \in T \text{ and } r, s \in \Gamma.$$

7 More notation and co-finite ideals

$$(7.1) \text{ Let } Der A(m) \text{ be the derivation algebra of } A(m). \text{ Let } e_1, \dots, e_n \text{ be} \\ \text{the standard basis of } \mathbb{C}^n \text{ and let } u = \sum u_i e_i \in \mathbb{C}^n. \text{ Let } D(u, r) = \\ \sum_{i=1}^n u_i t^r d_i, r \in \Gamma.$$

$$(7.2) \text{ From the earlier section } T \text{ can be identified with } V^1 \otimes A(m) \text{ where } V^1 \\ \text{can be identified with}$$

$$\bigoplus_{\substack{0 \leq k_i < m_i \\ 1 \leq i \leq n}} \mathbf{T}_k$$

$$(7.3) \text{ Now } D^0(m_0, m) \text{ is spanned by } t^r d_i, r \in \Gamma, 0 \leq i \leq n. \text{ Thus } D^0(m_0, m) \\ \text{can be identified with } Der A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$$

Z^0 can be identified with $\sum_{r \in \Gamma} \mathbb{C} t^r K_0$ as the rest of the space acts trivially on T . Thus $V^1 \otimes A(m)$ is an irreducible module for

$$L = L^0(\mathfrak{g}, \sigma) \oplus \text{Der} A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0 \oplus A(m),$$

where we identify $\sum_{r \in \Gamma} \mathbb{C} t^r K_0$ by $A(m)$ and $\frac{1}{c_0} t^r K_0$ goes to t^r which is well defined by 6.3.8.

(7.4) We note the following

$$\begin{aligned} t^r \cdot v \otimes t^s &= v \otimes t^{r+s}, \\ t^r d_0 \cdot v \otimes t^s &= \lambda(d_0) v \otimes t^{r+s} \text{ for } r, s \in \Gamma, v \in V^1. \end{aligned}$$

(7.5) Let $\mathring{\mathfrak{g}} = \{X \in \mathfrak{g} \mid \sigma_0 X = X, [h, X] = 0, h \in \mathfrak{h}(0)\}$

the following is easily checked.

(7.5.1) $\sigma_i(\mathring{\mathfrak{g}}) \subseteq \mathring{\mathfrak{g}}$ for $1 \leq i \leq n$.

(7.5.2) $\mathring{\mathfrak{g}} = \bigoplus_{\bar{k} \in \Lambda} \mathring{\mathfrak{g}}_{\bar{k}}$ is a natural

Λ - grading where $\mathring{\mathfrak{g}}_{\bar{k}} = \{X \in \mathring{\mathfrak{g}} \mid \sigma_i X = \xi_i^{k_i} X, 1 \leq i \leq n\}$

The corresponding multiloop algebra is denoted by

$$L(\mathring{\mathfrak{g}}, \sigma) = \bigoplus_{k \in \Lambda} \mathring{\mathfrak{g}}_{\bar{k}} \otimes t^k$$

It is clear that $L^0(\mathfrak{g}, \sigma) = L(\mathring{\mathfrak{g}}, \sigma)$.

When we say $X(k) = X \otimes t^k \in L(\mathring{\mathfrak{g}}, \sigma)$ we always mean $X \in \mathring{\mathfrak{g}}_{\bar{k}}$.

Thus $L \cong L(\mathring{\mathfrak{g}}, \sigma) \oplus \text{Der} A(m) \oplus A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$.

(7.6) The brackets in L are given as follows :

$$(7.6.1) \quad [X(k), Y(l)] = [X, Y](k+l),$$

$$(7.6.2) \quad [D(u, r), D(v, s)] = D(w, r+s) \text{ where } w = (u, s)v - (v, r)u,$$

$$(7.6.3) \quad [D(u, r), t^s] = (u, s)t^{r+s},$$

$$(7.6.4) \quad [D(u, r), X(k)] = (u, k)X(k+r),$$

$$(7.6.5) \quad [D(u, r), t^s d_0] = (u, s) t^{r+s} d_0.$$

Now we would like to classify the irreducible L - module $V^1 \otimes A(m)$.

We need some preparation for that.

(7.7) For $k \in \mathbb{Z}^n, X \in \mathfrak{g}_{\overline{k}}^\circ$ and $r_1, r_2, \dots, r_d \in \Gamma$, define

$$X(k, r_1, \dots, r_d) = X(k) - \sum X(k+r_i) + \sum_{i < j} X(k+r_i+r_j) + \dots + (-1)^d X(k+r_1+r_2+\dots+r_d)$$

Let F_d be the subspace of $L(\mathfrak{g}, \sigma)$ spanned by $X(k, r_1, \dots, r_d)$. It is easily checked that F_d is an ideal in $L(\mathfrak{g}, \sigma)$

(7.8) **Lemma**

$$(7.8.1) \quad F_d \subseteq F_{d-1}$$

$$(7.8.2) \quad [F_d, F_d] \subseteq F_{d+1}$$

Proof- Note that $X(k, r_1, \dots, r_d) = X(k, r_1, \dots, r_{d-1}) - X(k+r_d, r_1, \dots, r_{d-1})$

which proves (7.8.1).

Now consider for $l, k \in \mathbb{Z}^n, r_1, \dots, r_d, s \in \Gamma, X \in \mathfrak{g}_{\overline{k}}^\circ, Y \in \mathfrak{g}_{\overline{l}}^\circ$.

$$\begin{aligned} & [X(k, r_1, \dots, r_d), Y(l) - Y(l+s)] \\ &= [X, Y](k+l, r_1, \dots, r_d) \\ & - [X, Y](k+l+s, r_1, \dots, r_d) \\ &= [X, Y](k+l, r_1, r_2, \dots, r_d, s) \in F_{d+1}. \end{aligned}$$

By the above note we see that F_d is spanned by vectors $X(k) - X(k+s)$.

Thus from above (7.8.2) follows.

(7.9) In this subsection we recall some facts from [E2] on $Der A(m)$. Let

$$I(u, r) = D(u, r) - D(u, 0), u \in \mathbb{C}^n, r \in \Gamma \text{ It is easy to check.}$$

$$(7.9.1) \quad [I(u, r), I(v, s)] = (v, r)I(u, r) - (u, s)I(v, s) + I(w, s+r)$$

$$\text{where } w = (u, s)v - (v, r)u$$

Let I be the space spanned by $I(u, r), u \in \mathbb{C}^n, r \in \Gamma$ which can be seen as subalgebra of $Der A(m)$.

$$(7.10) \quad \text{For } d \geq 1, u \in \mathbb{C}^n, s_1, \dots, s_d, r \in \Gamma,$$

$$\text{Let } I_d(u, r, s_1, s_2, \dots, s_d) = I(u, r) - \sum_i I(u, r+s_i) + \sum_{i < j} I(u, r+s_i+s_j) - \dots + (-1)^d I(u, r+s_1+\dots+s_d)$$

$$s_j) + \cdots + (-1)^d I(u, r + s_1 + s_2 + \cdots + s_d)$$

Let I_d be the space spanned by $I_d(u, r, s_1, s_2, \cdots, s_d)$ for $u \in \mathbb{C}^n, r, s_1, \cdots, s_d \in \Gamma$.

The following is proved in [E2]

(7.11) **Lemma**

(7.11.1) I_d is a co-finite ideal in I .

(7.11.2) Any co-finite ideal of I contains I_d for large d .

(7.11.3) $I_1 = I$ and $I/I_2 \cong gl_n$.

8 Finite dimensional modules

(8.1) Let W be the subspace of $V^1 \otimes A(m)$ spanned by vectors of the form

$$t^r.v(s) - v(s) \text{ for } r, s \in \Gamma \text{ and } v \in V^1.$$

$$\text{Let } \tilde{L} = I \ltimes L(\mathfrak{g}, \sigma)$$

(8.2) **Lemma-** W is an $\tilde{L} \oplus A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$ module.

Proof- It is easy to check using the following

$$\begin{aligned} [D(u, r) - D(u, 0), X(k)] &= (u, k)(X(k + r) - X(k)), \\ [D(u, r) - D(u, 0), t^s] &= 0, \\ [L(\mathfrak{g}, \sigma), A(m)] &= 0, \\ [t^r d_0, A(m)] &= 0. \end{aligned}$$

Let $\tilde{V} = (V^1 \otimes A(m))/W$ which is an \tilde{L} -module. Notice that $A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$ acts as scalars on \tilde{V} and hence we ignore them. We would like to prove that \tilde{V} is completely reducible \tilde{L} -module.

(8.3) Recall that the Lie brackets in \tilde{L} are given by

$$(8.3.1) \quad [I(v, s), I(u, r)] = (u, s)I(v, s) - (v, r)I(u, r) + I(w, r + s),$$

$$\text{where } I(u, r) = D(u, r) - D(u, 0) \text{ and } w = (v, r)u - (u, s)v$$

$$(8.3.2) \quad [I(v, s), X(k)] = (v, k)(X(s + k) - X(k)),$$

$$(8.3.3) \quad [X(k), Y(l)] = [X, Y](k + l),$$

where $X \in \mathfrak{g}_{\bar{k}}, Y \in \mathfrak{g}_{\bar{l}}, k, l \in \mathbb{Z}^n, r, s \in \Gamma$ and $u, v \in \mathbb{C}^n$.

(8.4) Recall that we fixed $\lambda \in P(V)$. Let $\alpha_i = \lambda(d_i)$ Let $\alpha = \sum \alpha_i e_i \in \mathbb{C}^n$ and let V_1 is an \tilde{L} - module. Then we will define L - module structure on $L(V_1) = V_1 \otimes A_n$.

$$\begin{aligned} X(k) \cdot v_1 \otimes t^l &= (X(k)v_1) \otimes t^{l+k}, \\ D(u, r) \cdot v_1 \otimes t^l &= (I(u, r)v_1) \otimes t^{l+r} + (u, l + \alpha)v_1 \otimes t^{l+r}, \\ t^s v_1 \otimes t^l &= v_1 \otimes t^{s+l}, \\ t^r d_0 \cdot v_1 \otimes t^l &= \lambda(d_0) \cdot v_1 \otimes t^{l+r}, \end{aligned}$$

where $v_1 \in V_1, l, k \in \mathbb{Z}^n, r, s \in \Gamma$.

We need to check the brackets in (7.6). We will first check (7.6.4). Consider

$$\begin{aligned} D(u, r)X(k)(v_1 \otimes t^l) &= D(u, r)((X(k)v_1) \otimes t^{l+k}) \\ &= I(u, r) \cdot X(k)v_1 \otimes t^{l+k+r} + (u, l + k + \alpha)X(k)v_1 \otimes t^{l+k+r}. \end{aligned}$$

Consider

$$\begin{aligned} X(k)D(u, r)(v_1 \otimes t^l) &= X(k)(I(u, r)v_1 \otimes t^{l+r} + (u, l + \alpha)v_1 \otimes t^{l+r}) \\ &= X(k)I(u, r)v_1 \otimes t^{l+k+r} + (u, l + \alpha)X(k)v_1 \otimes t^{l+k+r}. \end{aligned}$$

Now we will use the fact $[I(u, r), X(k)] = (u, k)(X(k + r) - X(k))$.

$$\begin{aligned} \text{So } (D(u, r)X(k) - X(k)D(u, r)) \cdot (v_1 \otimes t^l) &= (u, k)(X(k + r) - X(k))v_1 \otimes t^{l+k+r} + (u, k)X(k)v_1 \otimes t^{l+k+r} \\ &= (u, k)X(k + r)v_1 \otimes t^{l+k+r} \\ &= (u, k)X(k + r)(v_1 \otimes t^l). \end{aligned}$$

This proves (7.6.4).

We will now check (7.6.2).

Consider

$$\begin{aligned} D(v, s)D(u, r)(v_1 \otimes t^l) &= D(v, s)(I(u, r)v_1 \otimes t^{l+r} + (u, l + \alpha)v_1 \otimes t^{l+r}) \\ &= I(v, s)I(u, r)v_1 \otimes t^{l+r+s} + (v, l + r + \alpha)I(u, r)v_1 \otimes t^{l+r+s} \\ &\quad + (u, l + \alpha)I(v, s)v_1 \otimes t^{l+r+s} + (u, l + \alpha)(v, l + r + \alpha)v_1 \otimes t^{l+r+s}. \end{aligned}$$

Similarly we have

$$\begin{aligned} D(u, r)D(v, s)(v_1 \otimes t^l) &= I(u, r)I(v, s)v_1 \otimes t^{l+r+s} \end{aligned}$$

$$\begin{aligned}
& +(u, l + s + \alpha)I(v, s)v_1 \otimes t^{l+r+s} \\
& +(v, l + \alpha)I(u, r)v_1 \otimes t^{l+r+s} \\
& +(v, l + \alpha)(u, l + s + \alpha)v_1 \otimes t^{l+r+s}.
\end{aligned}$$

Now we will use (8.3.1)

So

$$\begin{aligned}
& (D(v, s)D(u, r) - D(u, r)D(v, s))v_1 \otimes t^l \\
& = ((u, s)I(v, s) - (v, r)I(u, r) + I(w, r + s))v_1 \otimes t^{l+r+s} \\
& + (v, r)I(u, r)v_1 \otimes t^{l+r+s} \\
& - (u, s)I(v, s)v_1 \otimes t^{l+r+s} \\
& + (w, l + \alpha)v_1 \otimes t^{l+r+s} \\
& = D(w, r + s)(v_1 \otimes t^l)
\end{aligned}$$

This proves (7.6.2)

The remaining brackets 7.6.1, 7.6.3, 7.6.5 are trivial to verify.

(8.5) Recall

(8.5.1) T is an irreducible L -module from Proposition 6.2.

(8.5.2) \tilde{V} is an \tilde{L} -module from Lemma 8.2.

(8.5.3) $L(\tilde{V})$ is an L -module from (8.4).

We will now establish that T is contained in $L(\tilde{V})$ as L -modules.

For $v_k \in T_k$, let \bar{v}_k be the image in $\tilde{V} \cong T/W$.

Let $\tilde{\varphi}: T \rightarrow L(\tilde{V})$

$$\tilde{\varphi}(v_k) = \bar{v}_k \otimes t^k, k \in \mathbb{Z}^n.$$

(8.6) **Lemma** $\tilde{\varphi}$ is an L -module map.

Proof Consider $\tilde{\varphi}(D(u, r)v_k) = \overline{D(u, r)v_k} \otimes t^{k+r}$.

Now

$$\begin{aligned}
D(u, r)(\bar{v}_k \otimes t^k) &= \overline{(D(u, r) - D(u, 0))v_k} \otimes t^{k+r} \\
&+ (u, k + \alpha)\bar{v}_k \otimes t^{k+r} \\
&= \overline{D(u, r)v_k} \otimes t^{k+r}.
\end{aligned}$$

Thus we have verified $\tilde{\varphi}(D(u, r)v_k) = D(u, r)\tilde{\varphi}(v_k)$.

The rest of the relations are easy to verify. Clearly $\tilde{\varphi}$ is a non-zero map and since T is an irreducible L -module we have $T \subseteq L(\tilde{V})$ as L -module.

(8.7) **Theorem** \tilde{V} is completely reducible as \tilde{L} -module and all components are isomorphic.

We will prove some results before proving Theorem (8.7).

(8.8) **Proposition** Let \mathfrak{g} be a Lie algebra. Let V_1, V_2, \dots, V_n be mutually non-isomorphic irreducible \mathfrak{g} -modules. Suppose W is a non-zero \mathfrak{g} -submodule of $\bigoplus_{i=1}^n V_i$. Then there exists $S \subset \{1, 2, \dots, n\}$ such that

$$W = \bigoplus_{i \in S} V_i.$$

Proof Let $\tilde{\pi}_j: \bigoplus_{i=1}^n V_i \rightarrow V_j$ be the natural projection. Let $\pi_j = \tilde{\pi}_j|_W$. Let $S = \{j | \pi_j \neq 0\}$. Suppose $j \in S$, then $\pi_j(W) \neq 0$. Since V_j is irreducible, it follows that $\pi_j(W) = V_j$. Clearly

$$W \subset \bigoplus_{j \in S} V_j.$$

Claim

$$W = \bigoplus_{j \in S} V_j.$$

Note that the claim proves the proposition. We prove the claim by induction on n . Certainly the claim is true for $n = 1$. Let $w \in W$ and write $w = v_{i_1} + v_{i_2} + \dots + v_{i_k}$, where $0 \neq v_{i_j} \in V_{i_j}$. Then we define $l(w) = k$. We will now prove the claim for $n = 2$. Suppose $l(w) = 1$ for all $w \in W$, then clearly $W = V_1$ or $W = V_2$ and hence we are done. Suppose $l(w_0) = 2$ for some $w_0 \in W$. Then $w_0 = v_1 + v_2$ and $0 \neq v_i \in V_i$ for $i = 1, 2$. As noted earlier we have $\pi_1(W) = V_1$ and $\pi_2(W) = V_2$. Suppose $\ker \pi_1 = \ker \pi_2 = 0$, then $W = \pi_i(W) = V_i$ for $i = 1, 2$. This proves that $V_1 \cong V_2$ as \mathfrak{g} -modules, which is a contradiction. Now suppose $\ker \pi_1 \neq 0$. Then there exists $w \in \ker \pi_1$ and $w = v^1 + v^2$, $v^1 \in V_1$ and $v^2 \in V_2$. But $0 = \pi_1(w) = v^1$. Hence $v^2 \in W$. This proves $V_2 \subset W$. Recall that $w_0 = v_1 + v_2 \in W$. It now follows that

$v_1 \in W$ and hence $V_1 \subset W$. Thus $V_1 + V_2 = W$ and we are done. This completes the claim for $n = 2$.

Now assume the claim for $n - 1$ and we prove it for n . Note that $n \geq 3$. Suppose $S \subsetneq \{1, 2, \dots, n\}$. Then as noted earlier $W \subsetneq \bigoplus_{i \in S} V_i$ and by induction the claim follows. We can now assume that $S = \{1, 2, \dots, n\}$. Suppose $\ker \pi_i = 0$ for all i . Then $W = \pi_j(W) = V_j$ for all j . This proves that $V_j \cong V_i$ for all i and j , which is a contradiction to our assumption. We can now assume that $\ker \pi_i \neq 0$ for some i . Let $w \in \ker \pi_i$ and write $w = v_1 + v_2 + \dots + v_n, v_j \in V_j$. Now $v_i = \pi_i(w) = 0$. This proves $l(w) < n$. Let W_1 be the submodule generated by w . Then clearly $W_1 \subset \bigoplus_{j \neq i} V_j$. By induction it follows that there exists $T_1 \subsetneq \{1, 2, \dots, n\}$ such that $W_1 = \bigoplus_{j \in T_1} V_j \subsetneq W$. Let T_2 be the maximal subset of $\{1, 2, \dots, n\}$ such that $\bigoplus_{j \in T_2} V_j \subset W$. To prove the claim it is sufficient to prove $T_2 = \{1, 2, \dots, n\}$. So suppose there exists $j \notin T_2$. Since $\pi_j(W) \neq 0$, there exists $w \in W$ such that $\pi_j(w) \neq 0$. This proves $w \notin \bigoplus_{k \in T_2} V_k$. Now it is easy to find $w_2 \in W$ such that $w_2 = v_{i_1} + \dots + v_{i_l}$ and $0 \neq v_{i_j} \in V_{i_j}$ and $\{i_1, i_2, \dots, i_l\} \cap T_2 = \emptyset$. Now let W_2 be the submodule generated by w_2 . Note that $W_2 \subset \bigoplus_{j=1}^l V_{i_j}$. Now by induction we can find T_3 such that $T_3 \cap T_2 = \emptyset$ and $W_2 = \bigoplus_{k \in T_3} V_k$. Thus $\bigoplus_{i \in T_2 \cup T_3} V_i \subset W$ which contradicts the maximality of T_2 . Thus $T_2 = \{1, 2, \dots, n\}$ and proves the claim.

(8.9) **Lemma** \tilde{V} is graded irreducible \tilde{L} -module.

Proof Recall that \tilde{V} and \tilde{L} are Λ -graded. Further using the map $\tilde{\varphi}, T \cong \tilde{\varphi}(T)$ as L -modules. We will first note that the L -module generated by $\bar{v}_k \otimes t^k, k \in \mathbb{Z}^n$ is equal to the $\tilde{L} \oplus A(m)$ -module generated by $\bar{v}_k \otimes t^k, k \in \mathbb{Z}^n$. This follows from the fact

$$D(u, r)(\bar{v}_k \otimes t^k) = \overline{I(u, r)v_k} \otimes t^{k+r} + (u, k + \alpha)\bar{v}_k \otimes t^{k+r}$$

and

$$t^r \cdot \bar{v}_k \otimes t^k = \bar{v}_k \otimes t^{k+r}.$$

Also note that t^r acts trivially on \tilde{V} . From this it is easy to see that \tilde{V} is graded irreducible.

We will now prove a decomposition theorem for $L(\tilde{V})$ as L -module.

First we give some notation. Recall that

$$\tilde{V} = \bigoplus_{\bar{p} \in \Lambda} \tilde{V}_{\bar{p}}.$$

Let $p \in \mathbb{Z}^n$ and $\bar{p} \in \Lambda$. Define

$$L(\tilde{V})(\bar{p}) = \{\bar{v}_k \otimes t^{k+r+p}, \bar{v}_k \in \tilde{V}_{\bar{k}}, r \in \Gamma, k \in \mathbb{Z}^n\}.$$

Clearly $L(\tilde{V})(\bar{p})$ is closed under $A(m)$ and $\sum_{r \in \Gamma} \mathbb{C} t^r d_0$.

Consider for $X \in \mathfrak{g}_l$,

$$X(l) \cdot (\bar{v}_k \otimes t^{k+r+p}) = \overline{X(l)v_k} \otimes t^{k+l+r+p} \in L(\tilde{V})(\bar{p}).$$

$$D(u, s)(\bar{v}_k \otimes t^{k+r+p}) = \overline{I(u, s)v_k} \otimes t^{k+r+p+s} + (u, k+r+p+\alpha)\bar{v}_k \otimes t^{k+r+p+s}, s \in \Gamma.$$

We see that the above vector belongs to $L(\tilde{V})(\bar{p})$. Thus $L(\tilde{V})(\bar{p})$ is an L -module. Clearly

$$L(\tilde{V}) = \bigoplus_{\bar{p} \in \Lambda} L(\tilde{V})(\bar{p})$$

which is a finite sum of L -modules. We have seen already that $T \cong L(\tilde{V})(\bar{0})$ as L -modules and in particular $L(\tilde{V})(\bar{0})$ is an irreducible L -module.

(8.10) **Proposition** Each $L(\tilde{V})(\bar{p})$ is an irreducible L -module.

Proof Consider the map for a fixed $p \in \mathbb{Z}^n$ such that $\bar{p} \neq 0$,

$$\pi(\bar{p}) : L(\tilde{V})(\bar{p}) \rightarrow L(\tilde{V})(\bar{0})$$

$$\pi(\bar{p})(\bar{v}_k \otimes t^{k+r+p}) = \bar{v}_k \otimes t^{k+r}.$$

It is easy to see that $\pi(\bar{p})$ is a vector space isomorphism and not a L -module map. For example

$$(u, k+r+p+\alpha)\bar{v}_k \otimes t^{k+r+p} = d_i(\bar{v}_k \otimes t^{k+r+p}) \neq d_i(\bar{v}_k \otimes t^{k+r}) = (u, k+r+\alpha)\bar{v}_k \otimes t^{k+r}.$$

Now suppose W is a non-zero proper submodule of $L(\tilde{V})(\bar{p})$. Clearly $\pi(\bar{p})(W)$ is a non-zero proper subspace of $L(\tilde{V})(\bar{0})$. To prove that $L(\tilde{V})(\bar{p})$ is irreducible, it is sufficient to prove that $\pi(\bar{p})(W)$ is an L -module. Since $\pi(\bar{p})$ commutes with $L(\mathfrak{g}, \sigma) \oplus A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$, it follows that $\pi(\bar{p})(W)$ is a module for the above space. Since W is a weight module, it is easy to check that $\pi(\bar{p})(W)$ is also a weight module. Let $w = \bar{v}_k \otimes t^{k+r} \in \pi(\bar{p})(W)$ be a weight vector, then

$$D(u, s)(\bar{v}_k \otimes t^{k+r+p}) = \overline{I(u, s)v_k} \otimes t^{k+r+s+p} + (u, k+r+p+\alpha)\bar{v}_k \otimes t^{k+r+s+p} \in W.$$

Also

$$t^s(\bar{v}_k \otimes t^{k+r+p}) = \bar{v}_k \otimes t^{k+r+s+p} \in W.$$

This proves $\overline{I(u, s)v_k} \otimes t^{k+r+s+p} + (u, k+r+p+\alpha)\bar{v}_k \otimes t^{k+r+s+p} \in W$. This means $\pi^{-1}(\bar{p})(D(u, s)(\bar{v}_k \otimes t^{k+r})) \in W$. So $D(u, s)(\bar{v}_k \otimes t^{k+r}) \in \pi(\bar{p})(W)$. Thus $\pi(\bar{p})(W)$ is an L -module. This proves that $L(\tilde{V})(\bar{p})$ is irreducible L -module.

It is possible that some of the modules in $L(\tilde{V})$ are isomorphic. We need to develop the notion of graded automorphisms of \tilde{V} .

(8.11) **Definition** An \tilde{L} -module automorphism θ of \tilde{V} is called \bar{p} -graded if $\theta(\tilde{V}_{\bar{k}}) = \tilde{V}_{\bar{k}-\bar{p}}$ for all $\bar{k} \in \Lambda$.

(8.11.1) In this case $\dim \tilde{V}_{\bar{k}} = \dim \tilde{V}_{\bar{k}-\bar{p}}$.

(8.11.2) Suppose θ is a \bar{p} -graded automorphism of \tilde{V} . Choose minimal integer $N_p > 0$ such that $N_p \cdot \bar{p} = 0$ in Λ . Then clearly $\theta^{N_p} \tilde{V}_{\bar{k}} = \tilde{V}_{\bar{k}}$. Thus there exists a vector v in $\tilde{V}_{\bar{k}}$ such that $\theta^{N_p} v = \lambda v$ for some non-zero scalar λ . Consider the space

$$W = \{v \in \tilde{V} \mid \theta^{N_p} v = \lambda v\},$$

which can be seen as a graded submodule of \tilde{V} . Since \tilde{V} is graded irreducible by Lemma 8.9, we see that $W = \tilde{V}$. Thus $\theta^{N_p} = \lambda$ on \tilde{V} . Now by suitably multiplying θ by a scalar we can assume $\theta^{N_p} = 1$.

So here after we will work only with graded automorphisms of finite order. Recall we have fixed $\alpha \in \mathbb{C}^n$ from 8.4. For any L -module V , we define

$$V_k = \{v \in V \mid D(u, 0)v = (u, k + \alpha)v\},$$

for $k \in \mathbb{Z}^n$. Thus we have

$$L(\tilde{V})_k = \bigoplus_{\bar{p} \in \Lambda} L(\tilde{V})(\bar{p})_k.$$

(8.12) **Lemma** $\dim \tilde{V}_{\bar{k}} = \dim L(\tilde{V})(\bar{0})_k = \dim L(\tilde{V})(\bar{p})_{k+p}.$

Proof Proof follows from the definitions of $L(\tilde{V})(\bar{p})$.

(8.13) Suppose θ is a \bar{p} -graded automorphism of \tilde{V} and N_p be the order. Then θ defines an L -module isomorphism

$$\theta' : L(\tilde{V})(\bar{0}) \rightarrow L(\tilde{V})(\bar{p})$$

$$\theta'(\bar{v}_k \otimes t^{k+r}) = \theta(\bar{v}_k) \otimes t^{k+r+p}.$$

It is easy to check that θ' is an isomorphism of L -modules. Suppose there exists an L -module isomorphism θ' from $L(\tilde{V})(\bar{0})$ to $L(\tilde{V})(\bar{p})$, then k -weight vectors go to k -weight vectors under this isomorphism. Thus

$$\theta'(\bar{v}_k \otimes t^k) = \bar{w}_{k-p} \otimes t^k.$$

We now define $\theta(\bar{v}_k) = \bar{w}_{k-p}$. This can be checked to be \bar{p} -graded automorphism of \tilde{V} and can be assumed to be of finite order.

(8.14) **Proposition** There is a one-one correspondence between \bar{p} -graded \tilde{L} -module automorphism of \tilde{V} and isomorphism between $L(\tilde{V})(\bar{0})$ and $L(\tilde{V})(\bar{p})$.

Proof Proof follows from above discussion.

(8.15) **Lemma** Suppose such an automorphism exists, then

- (1) $\dim \tilde{V}_{\bar{k}} = \dim \tilde{V}_{\bar{k}-\bar{p}},$
- (2) $\dim L(\tilde{V})(\bar{0})_k = \dim L(\tilde{V})(\bar{0})_{k-p}.$

- (8.16) **Corollary** (1) $\dim \tilde{V}_{\bar{k}} = \dim \tilde{V}_{\overline{k+j\bar{p}}}, j \in \mathbb{Z}$.
 (2) $\dim L(\tilde{V})(j\bar{p})_{\overline{k+ip}} = \dim \tilde{V}_{\bar{k}}$ for any $i, j \in \mathbb{Z}$.

Proof Repeated application of θ . Note that θ^j is also an automorphism.

- (8.17) **Theorem** \tilde{V} is an \tilde{L} irreducible module if and only if $L(\tilde{V})(\bar{p}), \bar{p} \in \Lambda$ are mutually non-isomorphic as L -modules.

Proof We can suppose $L(\tilde{V})(\bar{0}) \cong L(\tilde{V})(\bar{p})$ for $0 \neq \bar{p} \in \Lambda$. Then there exists a \bar{p} -graded automorphism of \tilde{V} . Let N_p be the order. For $\bar{v}_k \in \tilde{V}_{\bar{k}}$, define

$$\bar{v}_k(0) = \bar{v}_k + \theta(\bar{v}_k) + \cdots + \theta^{N_p-1}(\bar{v}_k).$$

It is easy to check that $\theta(\bar{v}_k(0)) = \bar{v}_k(0)$. Let

$\tilde{M}_0 = \{\bar{v}_k(0), \bar{v}_k \in \tilde{V}_{\bar{k}}, \bar{k} \in \Lambda\}$, where $\{\}$ means the space generated by the vectors inside \tilde{V} . Clearly \tilde{M}_0 is a non-zero proper submodule of \tilde{V} .

This proves one side of the theorem. Now suppose $L(\tilde{V})(\bar{p}), \bar{p} \in \Lambda$ be mutually non-isomorphic modules. Suppose W is a \tilde{L} submodule of \tilde{V} . Then clearly

$$L(W) \subset L(\tilde{V}) = \bigoplus_{\bar{p} \in \Lambda} L(\tilde{V})(\bar{p}).$$

Then by Proposition 8.8, we see that there exists $S \subset \Lambda$ such that

$$L(W) = \bigoplus_{\bar{p} \in S} L(\tilde{V})(\bar{p}).$$

This means $L(\tilde{V})(\bar{p}) \subset L(W)$ for some \bar{p} . This means $\bar{v}_k \otimes t^{k+p} \in L(W)$, which means $\bar{v}_k \in W$. By Lemma 8.9, the module generated by \bar{v}_k is \tilde{V} . Hence $W = \tilde{V}$. This proves the other part of the theorem.

- (8.18) The aim is to find suitable irreducible \tilde{L} submodule of \tilde{V} .

Suppose \tilde{V} is irreducible, then we are done. If not then, $L(\tilde{V})(\bar{0}) \cong L(\tilde{V})(\bar{p})$. We will now find conditions for \tilde{M}_0 to be irreducible. Let ζ be the N_p th primitive root of unity. Define

$$\bar{v}_k(i) = \bar{v}_k + \zeta^i \theta(\bar{v}_k) + \cdots + \zeta^{(N_p-1)i} \theta^{N_p-1}(\bar{v}_k).$$

It is easy to check that $\theta(\bar{v}_k(i)) = \zeta^{-i} \bar{v}_k(i)$. Let

$$\tilde{M}_i = \{v \in \tilde{V} \mid \theta(v) = \zeta^{-i} v\},$$

which can be seen to be proper submodule of \tilde{V} . Further

$$(8.19) \quad \tilde{V} = \bigoplus_{i=0}^{N_p-1} \tilde{M}_i, \text{ which is an eigen space decomposition.}$$

(8.20) Define $M_i = \{\bar{v}_k(i) \otimes t^{k+r}, \bar{v}_k \in \tilde{V}_{\bar{k}}, r \in \Gamma\}$. It can be verified to be an L -module and further

$$M_i \subset \bigoplus_{j=1}^{N_p-1} L(\tilde{V})(jp).$$

Claim $\bar{v}_k(i) \otimes t^{k-jp+r} \in M_i$.

It is easy to check that

$$\bar{v}_k(i) = \zeta^i \theta(\bar{v}_k(i)) = \zeta^{ij} \theta^j(\bar{v}_k(i)) = \bar{w}_{k-jp}(i).$$

By definition $\bar{w}_{k-jp}(i) \otimes t^{k-jp+r} \in M_i$, thus $\bar{w}_{k-jp}(i) \otimes t^{k-jp+r} = \bar{v}_k(i) \otimes t^{k-j+r} \in M_i$. Hence the claim follows.

(8.20.1) In the Definition 8.20, one can replace Γ by $\Gamma_p = \Gamma + \mathbb{Z}_p$. Note that \mathbb{Z}_p is a finite subgroup of Λ .

(8.21) **Lemma** 1) $M_i \cong L(\tilde{V})(0)$ as L -module and in particular M_i is an irreducible L -module.

$$2) \quad \bigoplus_{i=0}^{N_p-1} M_i = \bigoplus_{j=0}^{N_p-1} L(\tilde{V})(jp).$$

Proof We know that $L(\tilde{V})(\bar{0}) \cong L(\tilde{V})(\bar{j}p)$ and M_i is contained in

$$\bigoplus_{j=0}^{N_p-1} L(\tilde{V})(jp). \text{ Thus } M_i \text{ is isomorphic to the sum of finitely many}$$

copies of $L(\tilde{V})(\bar{0})$. We will now compare the dimensions of the weight spaces.

Claim $\dim \tilde{V}_{\bar{k}} = \dim (M_i)_k$.

Consider the map $\bar{v}_k \rightarrow \bar{v}_k(i)$, which is clearly injective. The surjectivity is also obvious and hence the claim.

From Lemma 8.12, we know that $\dim \tilde{V}_{\bar{k}} = \dim L(\tilde{V})(\bar{0})_k$. This proves that the dimensions of the weight spaces of M_i and $L(\tilde{V})(\bar{0})$ are same.

Thus $M_i \cong L(\tilde{V})(\bar{0})$. In particular M_i is an irreducible L -module. It is easy to see that the sum at LHS is direct. Equality holds for dimensions reason. Hence both parts of the Lemma follows.

(8.22) Let \mathbb{Z}_p be the cyclic group generated by \bar{p} inside Λ . Let $\Lambda_p = \Lambda/\mathbb{Z}_p$. Consider

$$\mathbb{Z}^n \rightarrow \mathbb{Z}^n/\Gamma = \Lambda \rightarrow \Lambda_p.$$

Let Γ_p be the kernal of the above map.

(8.22.1) Note that each \tilde{M}_i is no more graded by Λ but by Λ_p . Similar to the proof of Lemma 8.9, one can prove that each \tilde{M}_i is Λ_p -graded irreducible.

Let $\bar{0} = q_0, q_1, \dots, q_{n_p-1}$ be a set of coset representative for Λ_p . Note that $n_p N_p = |\Lambda|$. Since θ is a \bar{p} -graded automorphism, we see that

$$L(\tilde{V})(q_i) \cong L(\tilde{V})(q_i + \bar{p}) \cdots \cong L(\tilde{V})(q_i + (N_p - 1)\bar{p}).$$

$$\text{Put } W(q_i) = \bigoplus_{j=1}^{N_p-1} L(\tilde{V})(q_i + j\bar{p}).$$

Define for $0 \leq i < N_p$, $0 \leq l < n_p$,

$$M_i^l = \{\bar{v}_k(i) \otimes t^{k+r+q_l}, \bar{v}_k \in \tilde{V}_{\bar{k}}, r \in \Gamma\}.$$

Note that $M_i^0 = M_i$.

This can be verified to be an L -module and $M_i^l \subset W(q_l)$.

Claim M_i^l is irreducible L -module and isomorphic to $L(\tilde{V})(q_l)$.

The proof is similar to the case $l = 0$. From the definition of $W(q_l)$, we know that M_i^l is isomorphic to sum of $L(\tilde{V})(q_l)$. As seen earliar for the case $l = 0$, $\dim (M_i^l)_{k+q_l} \cong \dim \tilde{V}_{\bar{k}}$, by considering the map $v_k \rightarrow v_k(i) \otimes t^{k+q_l}$. But by Lemma 8.12, we know that $\dim \tilde{V}_{\bar{k}} = \dim L(\tilde{V})(q_l)_{k+q_l}$. Thus

(8.23) $M_i^l \cong L(\tilde{V})(q_l)$, and in particular M_i^l is irreducible.

$$(8.24) \text{ Lemma } L(\tilde{M}_i) = \bigoplus_{l=0}^{n_p-1} M_i^l.$$

Proof Proof follows from the Definition in (8.4).

(8.25) We notice that R. H. S. in (8.23) is independent of i . Thus

$$L(\tilde{M}_i) \cong L(\tilde{M}_j).$$

Now we will record a simple fact.

(8.26) Let W_1 and W_2 be \tilde{L} -submodules of \tilde{V} . Then $W_1 \cong W_2$ as \tilde{L} -modules if and only if $L(W_1) \cong L(W_2)$ as L -modules. The above follows from the definitions. We now have the following.

(8.27) **Proposition** $\tilde{M}_i \cong \tilde{M}_j$.

Proof Proof follows from (8.26).

Now we have

$$L(\tilde{M}_i) = \bigoplus_{l=0}^{n_p-1} M_i^l,$$

where each M_i^l is irreducible L -module. This situation is similar to Theorem 8.17. We can now prove the following, whose proof is similar to Theorem 8.17.

(8.28) **Theorem** We fix i . \tilde{M}_i is irreducible as \tilde{L} -module if and only if $M_i^l, 0 \leq l < n_p$ are mutually non-isomorphic as L -modules.

Proof of the Theorem (8.7) Recall that $\tilde{V} = \bigoplus_{i=0}^{N_p-1} \tilde{M}_i$ and all components are isomorphic as \tilde{L} -modules, which follows from (8.19) and (8.27). Further $\dim \tilde{M}_i < \dim \tilde{V}$. Suppose \tilde{M}_i is reducible as \tilde{L} -module, then we can repeat the above process. This process has to stop for dimensions reasons. Hence the theorem follows.

(8.29) To avoid more complex notation we assume that each \tilde{M}_i is irreducible \tilde{L} -module.

9 Final Theorem

In this section we will describe the \tilde{L} -module structure of \tilde{M}_i . We are assuming that each \tilde{M}_i is irreducible \tilde{L} -module. We will actually prove that \tilde{M}_i is an irreducible module for the direct sum $gl_n \oplus \mathfrak{g}$. Recall that gl_n is a quotient

of I from 7.11.3 and $\overset{\circ}{\mathfrak{g}}$ is a quotient of $L(\overset{\circ}{\mathfrak{g}}, \sigma)$ by the map $X(k) \rightarrow X \in \overset{\circ}{\mathfrak{g}}_{\bar{k}}$. Here $X(\bar{k})$ is identified by X . We will start with a simple Lemma.

Suppose S is a vector space such that $S = \bigoplus_{\bar{k} \in \Lambda} S_{\bar{k}}$.

An operator $T : S \rightarrow S$ is called degree \bar{k} operator if $T(S_{\bar{l}}) \subseteq S_{\bar{l}+\bar{k}}$. The following lemma is trivial to see.

(9.1) **Lemma** Suppose T is a degree \bar{k} operator such that $\bar{k} \neq 0$. Suppose T acts as a scalar λ . Then $\lambda = 0$.

The following is well known. See Proposition 19.1(b) of [H].

(9.2) **Lemma** Let \mathfrak{g}' be a Lie algebra which need not be finite dimensional. Let (V_1, ρ) be an irreducible finite dimensional module for \mathfrak{g}' . We have a map $\rho : \mathfrak{g}' \rightarrow \text{End } V_1$. Then $\rho(\mathfrak{g}')$ is a reductive Lie-algebra with at most one dimensional center.

Let \mathfrak{g}_1 and \mathfrak{g}_2 be infinite dimensional Lie algebra such that \mathfrak{g}_1 acts on \mathfrak{g}_2 . Let $\mathfrak{g}' = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ be the natural semi-direct Lie algebra. Let J be an abelian ideal of \mathfrak{g}_2 which will not be assumed to be a ideal of \mathfrak{g}' .

(9.3) **Proposition** Suppose (V', ρ) is an irreducible finite dimensional module for \mathfrak{g}' . We have $\rho : \mathfrak{g}' \rightarrow \text{End}(V')$. Then $\rho(J)$ is central ideal in $\rho(\mathfrak{g}')$.

Proof From above Lemma (9.2), it follows that $\rho(\mathfrak{g}')$ is a reductive Lie algebra. Since \mathfrak{g}_2 is an ideal in \mathfrak{g}' we have $\rho(\mathfrak{g}_2)$ is an ideal in $\rho(\mathfrak{g}')$. Thus $\rho(\mathfrak{g}_2)$ is reductive Lie algebra. Now we know that J is abelian ideal in \mathfrak{g}_2 and hence $\rho(J)$ is contained in the center of $\rho(\mathfrak{g}_2)$. This proves $\rho(J)$ is actually contained in the center of $\rho(\mathfrak{g}')$. In particular $\rho(J)$ is an ideal in $\rho(\mathfrak{g}')$.

(9.4) **Theorem** \tilde{M}_i is actually an irreducible finite dimensional module for the direct sum $gl_n \oplus \overset{\circ}{\mathfrak{g}}$.

We need the following Lemma.

(9.4.1) **Lemma** Let $k \in \mathbb{Z}^n, r_1, r_2, \dots, r_d \in \Gamma$ and $d \geq 1$. Suppose $X(k, r_1, \dots, r_d)$ acts as a scalar λ on \tilde{V} . Then the scalar λ is zero.

Proof First we note that if $X \in \mathring{\mathfrak{g}} \cap \mathfrak{g}(\overline{\sigma}, \overline{\sigma})$ then $X \in \mathfrak{h}(0)$. This follows from the theory of finite dimensional simple Lie algebras. Suppose $\overline{k} = 0$, then $X \in \mathfrak{h}(0)$. From 6.3.5, it follows that $X(k, r) = X(k) - X(k+r)$ is zero on \tilde{V} . Since $X(k, r_1, r_2, \dots, r_d)$ is spanned by $X(k, r), r \in \Gamma$, it follows that λ is zero. Now suppose $\overline{k} \neq 0$, then from Lemma 9.1, it follows that λ is zero. This completes the proof of the Lemma.

Proof of Theorem 9.4 Let ρ denote the \tilde{L} -module action on \tilde{M}_i and note that ρ is independent of i as all \tilde{M}_i are isomorphic. Consider $(\ker \rho) \cap I$ which is a co-finite ideal of I . Thus from 7.11.2, it follows that $(\ker \rho) \cap I$ contains I_d for large d . Consider

$$[I(u, k, r_1, r_2, \dots, r_d), X(l)] = X(k+l, r_1, r_2, \dots, r_d) \in \ker \rho. \quad (*)$$

This proves F_d acts trivially on \tilde{M}_i . But we know that $[F_{d-1}, F_{d-1}] \subseteq F_d$ by 7.8.2. Thus $\rho(F_{d-1})$ is an abelian ideal in $\rho(L(\mathring{\mathfrak{g}}, \sigma))$. By Proposition 9.3 it follows that $\rho(F_{d-1})$ is a central ideal in $\rho(\tilde{L})$. It is well known that center acts as scalars on a finite dimensional irreducible module. Thus $X(k, r_1, r_2, \dots, r_d)$ acts as scalar on each \tilde{M}_i and the scalar is independent of i . Thus $X(k, r_1, r_2, \dots, r_d)$ acts as a single scalar on \tilde{V} . Now by Lemma 9.4.1, scalar is zero. Thus F_{d-1} acts trivially on each \tilde{M}_i . It now follows that $\rho(I_{d-1})$ is an ideal in $\rho(\tilde{L})$ by (*). By Lemma 4.2 of [E2], I_{d-1} acts trivially. By repeating this argument finitely many times we see that I_2 acts trivially and F_2 acts trivially. Now from above argument we see that F_1 acts trivially. Thus $I_2 \oplus F_1$ acts trivially on \tilde{M}_i . As noted in the beginning of the section we see that each \tilde{M}_i is a module for $gl_n \oplus \mathring{\mathfrak{g}}$ and hence the theorem is proved.

(9.5) By Lemma 2.7 of [HL], there exists an irreducible module \tilde{V}_1 of gl_n and an irreducible module \tilde{V}_2 for $\mathring{\mathfrak{g}}$ such that $\tilde{V}_1 \otimes \tilde{V}_2 \cong \tilde{V}$ as $gl_n \oplus \mathring{\mathfrak{g}}$ -module. Regarding gradation, note that all vectors of I are grade zero and hence we can assume \tilde{V}_1 is zero graded and \tilde{V}_2 is Λ_p -graded.

(9.6) We will now describe L -module T in terms of \tilde{V}_1 and \tilde{V}_2 .

Let $\{E_{ij}\}_{1 \leq i, j \leq n}$ be the standard basis of gl_n . From [E2] it follows that

$I(u, r) = \sum u_i r_j m_j E_{ji} \in gl_n$ is linear in both variables mod I_2 .

Let $u = \sum u_i e_i$ and $r = \sum m_j r_j e_j$,

then $D(u, r) = D(u, 0) + \sum_{i,j} u_i r_j m_j E_{ji} \text{ mod } I_2$.

Let $\tilde{V}_2 = \bigoplus_{\bar{k} \in \Lambda_p} \tilde{V}_{2, \bar{k}}$ be the Λ_p -gradation.

We will now define L -module on $\tilde{V}_1 \otimes \tilde{V}_2 \otimes A_n$.

$$\begin{aligned} D(u, r)(v_1 \otimes v_2 \otimes t^k) &= (u, k + \alpha)v_1 \otimes v_2 \otimes t^{k+r} \\ &\quad + \sum (u_i r_j m_j E_{ji} v_1) \otimes v_2 \otimes t^{k+r}, \\ X(l)(v_1 \otimes v_2 \otimes t^k) &= v_1 \otimes Xv_2 \otimes t^{k+l}, \\ t^r \cdot v_1 \otimes v_2 \otimes t^k &= v_1 \otimes v_2 \otimes t^{k+r}, \\ t^r d_0 \cdot v_1 \otimes v_2 \otimes t^k &= \lambda(d_0)v_1 \otimes v_2 \otimes t^{k+r}. \end{aligned}$$

From the above discussion we see that this module is isomorphic to $L(\tilde{M}_i)$ as L -module.

Let $\bigoplus_{\underline{k} \in \Lambda_p} \tilde{V}_{2, \underline{k}} = \tilde{V}_2$ be the Λ_p -gradation.

Then consider the submodule of $L(\tilde{M}_i)$

$\bigoplus_{k \in \mathbb{Z}^n} \tilde{V}_1 \otimes \tilde{V}_{2, \underline{k}} \otimes t^k$ which is easy to see that it is isomorphic to M_i for any i as L -module and hence irreducible L -module. By defining $t^r K_p$ to be zero for $1 \leq p \leq n$ we see that the above module is actually a τ^0 -module.

(9.6.1) We have seen that $T \cong L(\tilde{V})(0)$ as L -modules from (8.6). But $M_i \cong L(\tilde{V})(0)$ by 8.21. Thus it follows that $M_i \cong T$ as L -modules for each i . Note that it is difficult to give direct module map between M_i and T as there is a twisting taking place. Thus we described T explicitly as L -module. Let $M = \text{Ind}_{\tau_0 + \tau^+}^\tau T$. Then there exists a unique maximal submodule M^{rad} intersecting T trivially. Thus M/M^{rad} is irreducible and isomorphic to the original module V .

(9.7) **Theorem** Let V be an irreducible integrable module for τ with K_0 acts as $C_0 > 0$ and K_i acts trivially. Let T and M as above. Then $V \cong M/M^{rad}$ as τ -modules.

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